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SOME FIXED POINT AND COINCIDENCE POINT RESULTS IN PARTIAL METRIC SPACES

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ABSTRACT: In this paper, we consider fixed point theorems on partial metric spaces. We studied weakly compatible functions on partial metric spaces, and we prove that under some conditions these functions have a unique point of coincidence and common fixed point.

Keywords: Common fixed point, Coincidence point, Complete metric, Contraction, Partial metric, weakly compatible.

INTRODUCTION

It is well known that the Banach contraction principle is a very useful, simple and classical tool in nonlinear analysis. There exist a vast literature concerning its various generalizations and extensions. In (6), Matthews extended the Banach contraction mapping theorem to the partial metric context for applications in program verification. After that, fixed-point results in partial metric spaces have been studied by Ran and Reurings (7). Recently, Abbas and Jungck (1), have studied common fixed point results for non-commuting mappings without continuity in cone metric space with normal cone, and a great deal of new results in this notion published in (3, 4, 5, 8).
First, we recall some definitions of partial metric spaces and some their properties.

Definition 1.1

A partial metric space on nonempty set X is a function p:
$X \times X \longrightarrow R^+$ such that for all x, y, z $\in$ X:
(p1) x = y $\Leftrightarrow$ p(x, y) = p(x, y) = p(y, y);
(p2) p(x, x) $\leq$ p(x, y);
(p3) p(x, y) = p(y, x);
(p4) p(x, y) $\leq$ p(x, z) + p (z, y) − p (z, z).
A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X. It is clear that if p(x, y) = 0, then from (p1), (p2), and (p3), we conclude that x = y. But if x = y, p(x, y) may not be 0. If p is a partial metric on X, then the function p: $X \times X \longrightarrow R^+$, given by
$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$,

Is a metric on X.

Definition 1.2

Let (X, p) be a partial metric space. Then
(i) a sequence $\{x_n\}$ in partial metric space (X, p) converges to a point x $\in$ X

if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$;
(ii) a sequence $\{x_n\}$ in partial metric space (X, p is called cauchy sequence if

there exists(and is finite) if $\lim_{n,m\to\infty} p(x_m, x_n)$;
(iii) a partial metric space (X, p) is said to be complete if every Cauchy sequence

$\{x_n\}$ in X converges to be $x \in X$, that is $p(x, x) = \lim_{n,m\to\infty} p(x_m, x_n)$
Now; we recall the following technical Lemma (see [2] and [6]).

Lemma 1.3

Let (X, p) be a partial metric space, then
(a) $\{x_n\}$ is a Cauchy sequence is (X, p) if and only if it is a Cauchy sequence in the metric space (X, p).
(b) a partial metric space (X, p) is complete if only if the metric space (X, p^s)

is complete; furthermore $\lim_{n\to\infty} p^s(x_n, x) = 0$ and if only if

$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Definition 1.4. A point $u \in X$ is called a coincidence point of the pair f, g and v is its point of coincidence if fu = gu = v. The pair (f, g) is said to be weakly compatible if for each $x \in X$, f x = gx implies that f gx = g f x.

2. Main results

Theorem 2.1

Let (X, p) be a compleat partial metric space and the mappings $f, g, h : X \to X$ satisfy:

$p(fx, gy) \le \alpha_1 p(hx, hy) + \alpha_2 p(hx, fx) + \alpha_3 p(hy, gy) + \alpha_4 p(hx, gy) + \alpha_5 p(hy, fx)$, for all $x, y \in X$, where α_i are nonnegative and $\sum_{i=1}^{5} \alpha_i \le 1. If$ f(X) Ug(X) $\subset$ h(X) and h(X) is a complete subspace of X, then f, g, and h have a unique point of coincidence. Moreover, if f, h and g, h are weakly compatible, then f, g, and have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary. By using the condition f(X) Ug(X) $\subset$ h(X), choose a sequence $\{x_n\}$ such that $hx_{2n+1} = fx_{2n}$ and $hx_{2n+2} = gx_{2n+1}$ for all
$n \in N$. Applying contractive condition we obtain that

$p(hx_{2n}+1, hx_{2n+2}) = p(fx_{2n}, gx_{2n+1}) \le \alpha_1 p(hx_{2n}, hx_{2n+1}) + \alpha_2 p(hx_{2n}, hx_{2n+1})$
$+ \alpha_3 p(hx_{2n+1}, hx_{2n+2}) + \alpha_4 p(hx_{2n}, hx_{2n+2}) + \alpha_5 p(hx_{2n+1}, hx_{2n+1}))$
$\le \alpha_1 p(hx_{2n}, hx_{2n+1}) + \alpha_2 p(hx_{2n}, hx_{2n+1})$
$+ \alpha_3 p(hx_{2n+1}, hx_{2n+2}) + \alpha_4 p(hx_{2n}, hx_{2n+1})$
$+ \alpha_4 p(hx_{2n+1}, hx_{2n+2}) - \alpha_4 p(hx_{2n+1}, hx_{2n+1}) + \alpha_5 p(hx_{2n+1}, hx_{2n+1}))$
$\le \alpha_1 p(hx_{2n}, hx_{2n+1}) + \alpha_2 p(hx_{2n}, hx_{2n+1})$
$+ \alpha_3 p(hx_{2n+1}, hx_{2n+2}) + \alpha_4 p(hx_{2n}, hx_{2n+1})$
$+ \alpha_4 p(hx_{2n+1}, hx_{2n+2}) + \alpha_5 p(hx_{2n+1}, hx_{2n+1}))$ (2.1)

It follows that $(1 - \alpha_3 - \alpha_4 - \alpha_5) p(hx_{2n+1}, hx_{2n+2}) \le (\alpha_1 - \alpha_2 - \alpha_4) p(hx_{2n}, hx_{2n+1})$.
That is

$$p(hx_{2n+1}, hx_{2n+2}) \le \left(\frac{\alpha_1 + \alpha_2 + \alpha_4}{1 - \alpha_3 - \alpha_4 - \alpha_5} \right) p(hx_{2n}, hx_{2n+1}), \quad (2.2)$$

and similarly

$p(hx_{2n+2}, hx_{2n+3}) = p(fx_{2n+1}, gx_{2n+2}) \leq \alpha_1 p(hx_{2n+1}, hx_{2n+2})$
$+ \alpha_2 p(hx_{2n+1}, hx_{2n+2}) + \alpha_3 p(hx_{2n+2}, hx_{2n+3})$
$+ \alpha_4 p(hx_{2n+1}, hx_{2n+3}) + \alpha_5 p(hx_{2n+2}, hx_{2n+2}))$
$\_\ \alpha_1 p(hx_{2n+1}, hx_{2n+2}) + \alpha_2 p(hx_{2n+1}, hx_{2n+2})$
$+ \alpha_3 p(hx_{2n+2}, hx_{2n+3}) + \alpha_4 p(hx_{2n+1}, hx_{2n+2})$
$+ \alpha_4 p(hx_{2n+2}, hx_{2n+3}) - \alpha_4 p(hx_{2n+2}, hx_{2n+2}) + \alpha_5 p(hx_{2n+2}, hx_{2n+3}))$
$\_\ \alpha_1 p(hx_{2n+1}, hx_{2n+2}) + \alpha_2 p(hx_{2n+1}, hx_{2n+2})$
$+ \alpha_3 p(hx_{2n+2}, hx_{2n+3}) + \alpha_4 p(hx_{2n+1}, hx_{2n+2})$
$+ \alpha_4 p(hx_{2n+2}, hx_{2n+3}) + \alpha_5 p(hx_{2n+2}, hx_{2n+3}))$. (2.3)

It follows that $(1-\alpha_3-\alpha_4-\alpha_5) p(hx_{2n+2}, hx_{2n+3}) \leq (\alpha_1+\alpha_2+\alpha_4) p(hx_{2n+1}, hx_{2n+2})$, that is

$$p(hx_{2n+2}, hx_{2n+3}) \leq \left(\frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3-\alpha_4-\alpha_5}\right) p(hx_{2n+1}, hx_{2n+2})$$
$$\leq \left(\frac{\alpha_1+\alpha_2+a_4}{1-\alpha_3-\alpha_4-\alpha_5}\right)^2 p(hx_{2n}, hx_{2n+1}) \quad (2.4)$$

Now, from (2.2) and (2.4) by induction, we obtain that

$$p(hx_{2n+1}, hx_{2n+2}) \leq \left(\frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3-\alpha_4-\alpha_5}\right) p(hx_{2n}, hx_{2n+1})$$
$$\leq \left(\frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3-\alpha_4-\alpha_5}\right)^2 p(hx_{2n-1}, hx_2 n) \quad (2.5)$$
$$\leq \left(\frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3-\alpha_4-\alpha_5}\right)^3 p(hx_{2n-2}, hx_{2n-1})$$
$$\leq \ldots \leq \left(\frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3-\alpha_4-\alpha_5}\right)^{n+3} p(hx_0, hx_1)$$

On the other hand.

$$(2.6) \qquad p(hx_{2n+2}, hx_{2n+3}) \leq \left(\frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3-\alpha_4-\alpha_5}\right) p(hx_{2n+1}, hx_{2n+2})$$
$$\leq \ldots \leq \left(\frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3-\alpha_4-\alpha_5}\right)^{n+4} p(hx_0, hx_1)$$

Let $\alpha = \left(\frac{\alpha_1+\alpha_2+\alpha_4}{1-\alpha_3-\alpha_4-\alpha_5}\right)$ Since

$p(hx_{2n+1}, hx2m+1) \leq p(hx_{2n+1}, hx_{2n+2}) + p(hx_{2n+2}, hx_{2m+1}) - p(hx_{2n+1}, hx_{2n+1})$,
and
$p(hx_{2n+2}, hx_{2m+1}) \leq p(hx_{2n+2}, hx_{2n+3}) + p(hx_{2n+3}, hx_{2m+1}) - p(hx_{2n+3}, hx_{2n+3})$.
Then for every $n < m$, we have

$$p(hx_{2n+1}, hx_{2m+1}) \leq p(hx_{2n+1}, hx_{2n+2}) + \ldots + p(hx_{2n}, hx_{2m+1})$$
$$\leq (\alpha^{n+3}+\alpha^{n+4}+\ldots+\alpha^m) p(hx_0, hx_1) \quad (2.7)$$
$$\leq \alpha^n (1+\alpha+\alpha^2+\ldots+\alpha^n) p(hx_0, hx_1)$$

$$= \alpha^n \left(\frac{1-\alpha^{n+1}}{1-\alpha} \right) p(hx_0, hx_1),$$

and similarly

$$p(hx_{2n}, hx_{2m+1}) \leq \alpha^n \left(\frac{1-\alpha^{n+1}}{1-\alpha} \right) p(hx_0, hx_1)$$

$$p(hx_{2n}, hx_{2m}) \leq \alpha^n \left(\frac{1-\alpha^{n+1}}{1-\alpha} \right) p(hx_0, hx_1)$$

and (2.8)

$$\mathrm{p}(\mathrm{hx}_n, hx_m) \leq \alpha^n \left(\frac{1-\alpha^{n+1}}{1-\alpha} \right) p(hx_0, hx_1)$$

Hence for n < m

$$p(hx_n, hx_m) \leq \alpha^n \left(\frac{1-\alpha^{n+1}}{1-\alpha} \right) p(hx_0, hx_1) \qquad (2.9)$$

Where $\alpha \to 0$ as n $\to \infty$. Thus lim $_{n\to\infty}$ p(hx$_n$, hx$_m$) = 0. By definition of p^s, we have p^s(x, y) $\leq$ p(x, y), so lim $_{n\to\infty}$ p^s(hx, hxm) = 0. Then by Lemma 1.3, we conclude that {hx$_n$} is a Cauchy sequence in (p^s,X). Since the subspace h(X) is complete, there exist u, v $\in$ X such that hx$_n$ $\to$ v = hu(n $\to \infty$). We will prove that hu = fu = gu. Firstly, let us estimate that p(hu, fu) = p(v, fu). We have that

$$p(hu, fu) \leq p(hu, hx_{2n+2}) + p(hx_{2n+2}, fu) - p(hx_{2n+2}, hx_{2n+2})$$
$$\leq p(hu, hx_{2n+2}) + p(hx_{2n+2}, fu) = p(v, hx_{2n+2}) + p(gx_{2n+1}, fu) \qquad (2.10)$$

On the other hand

$$p(fu, gx_{2n+1}) \leq \alpha_1 p(hu, hx_{2n+1}) + \alpha_2\, 2p(hu, fu)$$
$$+ \alpha_3\, p(hx_{2n+1}, gx_{2n+1}) + \alpha_4\, p(hu, gx_{2n+1}) + \alpha_5\, p(hx_{2n+1}, fu))$$
$$= \alpha_1\, p(v, fx2n) + \alpha_2\, p(v, fu) + \alpha_3\, p(fx_{2n}, gx_{2n+1})$$
$$+ \alpha_4\, p(v, gx_{2n+1}) + \alpha_5\, p(fx_{2n}, fu)). \qquad (2.11)$$

It follows by (2.10)

$$p(fu, v) \leq p(u, hx_{2n+2}) + p(gx_{2n+1}, fu)$$
$$\leq p(u, gx_{2n+1}) + (\alpha_1 + \alpha_4) p(u, fx_{2n}) + (\alpha_2 + \alpha_4) p(v, fu)$$
$$+ \alpha_3 p(fx_{2n}, gx_{2n+1}) + \alpha_4 p(u, gx_{2n+1}) + \alpha_5 p(fx_{2n}, fu). \qquad (2.12)$$

That is

$$(1 - \alpha_2 - \alpha_4)\, p(fu, v) \leq (\alpha_1 + \alpha_4) p(v, fx_{2n}) + (1 + \alpha_4) p(v, gx_{2n+1})$$
$$+ \alpha_3\, p(fx_{2n}, gx_{2n+1}) + \alpha_5\, p(fx_{2n}, fu). \qquad (2.13)$$

Then

$$p(v, fu) \leq \left(\frac{\alpha_1 + \alpha_4}{1-\alpha_2-\alpha_4} \right) p(v, fx_{2n}) + \left(\frac{1-\alpha_4}{1-\alpha_2-\alpha_4} \right) p(v, gx_{2n+1})$$
$$+ \left(\frac{\alpha_3}{1-\alpha_2-\alpha_4} \right) p(fx_{2n}, gx_{2n+1}) + \left(\frac{\alpha_5}{1-\alpha_2-\alpha_4} \right) p(fx_{2n}, fu). \qquad (2.14)$$

Since p(v, gx$_{2n+1}$) = lim $_{n\to\infty}$ p(hx$_n$, hx$_{2n+2}$) = 0, so similarly we can show that

lim $_{n\to\infty}$ p(v, fx_{2n}) = 0, lim $_{n\to\infty}$ p(fu, fx_{2n}) = 0, and lim $_{n\to\infty}$ p(gx_{2n+1}, fx_{2n}) = 0.
By gathering of later obtained results we have p(v, fu) = 0, that is fu = hu = v.

Similarly using

$$\begin{aligned} p(hu,gu) &\le p(hu,hx_{2n+1}) + p(hx_{2n+1},gu) - p(hx_{2n+1},hx_{2n+1}) \\ &\le p(hu,hx_{2n+1}) + p(hx_{2n+1},gu) \\ &= p(hu,hx_{2n+1}) + p(fx_{2n},gu) \qquad (2.15) \end{aligned}$$

it can be deduced that hu = gu = v. It follows that v is a common point of coincidence for f, g, h, that is
v = fu = gu = hu.
Now we prove that the point of coincidence of f, g, h is unique. Suppose that there is another point v1 2 X such that
$v_1 = fu_1 = gu_1 = hu_1$,
for some $u_1 \in$ X. Using the contractive condition we obtain that

$$\begin{aligned} p(v,v_1) &= p(fu,gu_1) + \alpha_1 p(hu,hu_1) + \alpha_2 p(hu,fu) \\ &+ \alpha_3 p(hu_1,gu_1) + \alpha_4 p(hu,gu_1) + \alpha_5 p(hu_1,fu) \\ &= \alpha_1 p(v,v_1) + \alpha_2 p(v,v) + \alpha_3 p(v_1,v_1) + \alpha_4 p(v,v_1) + \alpha_5 p(v_1,v) \\ &\le \alpha_1 p(v,v_1) + \alpha_2 p(v,v_1) + \alpha_3 p(v_1,v) + \alpha_4 p(v,v_1) + \alpha_5 p(v_1,v) \\ &= (\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5) p(v,v_1) \end{aligned}$$

Since ($\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$) $\le$ 1 it follows that p(v, v_1) = 0, that is, v = v_1.
Using weak compatibility of the pairs (f, h) and (g, h) and Proposition 1.12 of [1], it follows that the mappings f, g, h have a unique common fixed point, that is, fv = gv = hv = v.
Now; by above Theorem we have the following results.

Corollary 2.2

Let (X, p) be a partial metric space and the mappings f, g, h : X $\to$ X satisfy
p(fx, gy) $\le \alpha p$ (hx, hy) + β [(hx, fx) + p(hy, gy)] + γ [p(hx, gy) + p(hy, fx)],for all x, y $\in$ X, where $\alpha, \beta, \gamma \ge 0$ and $\alpha + 2\beta + 2\gamma$ < 1. If f(X) Ug(X) $\subset$ h(X)
and h(X) is a complete subspace of X, then f, g, and h have a unique point of coincidence. Moreover, if (f, g) and (g, h) are weakly compatible, then f, g, and h have a unique common fixed point.

Corollary 2.3

Let (X, p) be a complete partial metric space, and let the mappingsf, g :X $\to$ X satisfy
p(fx, gy) $\le \alpha$ p(x, y) β [p(x, fx), p(y, gy)] + γ [p(x, gy) + p(y, fx)],
for allx, y $\in$ X, where $\alpha, \beta, \gamma \ge$ and $\alpha + 2\beta + 2\gamma$ < 1. Then f and g have a unique common fixed point inX. Moreover, any fixed point of f is a fixed point of g, and conversely.

Corollary 2.4. Let (X, p) be a partial metric space, and let f, g : X $\to$ X be such thatf(X) $\subset$ g(X). Suppose that
p(fx, fy) $\le \alpha$ p(fx, gx) + β p(fy, gy) + γ p(gx, gy)
for all x, y $\in$ X, where $\alpha, \beta, \gamma \in$ [0, 1] and $\alpha + \beta + \gamma$ < 1, and let fx = gx imply that fgx = ggx for each x $\in$ X. If f(X) or g(X) is a complete subspace of X, then the mappings f and g have a unique common fixed point in X.

Example 2.5

Let p : X × X $\rightarrow$ [0, ∞) by p(x, y) = max{x, y}.Then (X, p) is complete because (X, dp) is complete. Indeed for any x, y $\in$ X.

dp(x, y) = 2p(x, y) − p(x, x) − p(y, y) = 2max{x, y} − (x + y) = |x − y|

Thus (X, dp) = ([0,+ ∞), |.|) is the usual metric space, which complete. We define fx = $\frac{x}{8}$, gx = $\frac{x}{12}$ and hx = $\frac{x}{2}$ then p(hx, hy) = $\frac{x}{2}$; p(hx, fx) = $\frac{x}{2}$; p(hy, gy) = $\frac{y}{2}$; p(hx, gy) = $\frac{x}{2}$; p(hy, fx) = $\frac{y}{2}$;

and p(fx, gy) = $\frac{x}{8}$

p(fx, gy) = $\frac{x}{8} \leq \alpha_1$ p(hx, hy) + α_2 p(hx, fx) + α_3 p(hy, gy) + α_4 p(hx, gy) + α_5 p(hy, fx)

$$= \alpha_1 \frac{x}{2} + \alpha_2 \frac{x}{2} + \alpha_3 \frac{y}{2} + \alpha_4 \frac{x}{2} + \alpha_5 \frac{x}{2}$$

$$\leq (\sum_{i=1}^{5} \alpha_i) \frac{x}{2} \leq \frac{x}{2}$$

Hence all the conditions of Theorem (2.1) are satisfied. Therefor x = 0 is common fixed point of the mappings f, g, and h.

Example 2.6

Similarly of example (2.5) let p : X × X $\rightarrow$ [0, ∞) by p(x, y) = max{x, y}. We define fx = $-x^2$, gx = x^2 and hx = x^3 then
p(hx, hy) = x^3; p(hx, fx) = x^3; p(hy, gy) = y^3; p(hx, gy) = x^3; p(hy, fx) = x^3;
and p(fx, gy) = y^2

p(fx, gy) = $y^2 \leq \alpha_1$ p(hx, hy) + α_2 p(hx, fx) + α_3 p(hy, gy) + α_4 p(hx, gy) + α_5 p(hy, fx)

$$= \alpha_1 x^3 + \alpha_2 x^3 + \alpha_3 y^3 + \alpha_4 y^3 + \alpha_5 x^3$$

$$\leq (\sum_{i=1}^{5} \alpha_i) y^3 \leq x^3$$

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Hence all the conditions of Theorem (2.1) are satisfied. Therefor x = 0 is common fixed point of the mappings f, g, and h.

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